

Even Dimensional Improper Affine Spheres

Marcos Craizer and Pedro de M. Rios

Abstract. In this paper, we study even dimensional improper affine spheres. The main emphasis is given in two classes of examples that generalize the 2-dimensional improper spheres. The generalization of the indefinite Blaschke metric case is related to the center-chord transformation of a pair of Lagrangian submanifolds, while the generalization of the definite case is related to special Kähler manifolds. Although most of the paper deals with these two classes, we also present some examples of improper affine spheres that do not belong to them.

Mathematics Subject Classification (2010). 53A15.

Keywords. Parabolic affine spheres, Special Kähler Manifolds, Center-chord transform.

1. Introduction

An immersion $\phi : M^m \rightarrow \mathbb{R}^{m+1}$ whose Blaschke normal vectors are pointing to a center is called an affine sphere. When the center is at infinity, i.e., the Blaschke normal vectors are parallel, the immersion is called an improper affine sphere. This class of manifolds is quite large and has been studied by various researchers ([8]). In this paper we shall study even dimensional improper affine spheres.

We shall consider immersions $\phi : M^{2n} \rightarrow \mathbb{R}^{2n} \times \mathbb{R}$ transversal to $\xi = (0, 1)$. Denoting $\phi(r) = (x(r), f(r))$, we have that the derivative of x , $Dx(r)$ is invertible, for any $r \in M$. Denote the canonical symplectic form in \mathbb{R}^{2n} by ω and let $c(r)$ be the symplectic gradient of f , i.e.,

$$Df(r)u = \omega(Dx(r)u, c), \quad u \in T_r M. \quad (1.1)$$

The first author thanks CNPq and the second author Fapesp for financial support during the preparation of this manuscript.

Define the linear transformation $A(r)$ of $T_r M$ by

$$Dc(r) = Dx(r) \cdot A(r). \quad (1.2)$$

We shall show in this paper that ϕ is an improper affine sphere if and only if the determinant of $A(r)$ does not depend on r . When the linear transformation A itself does not depend on r , we shall call the immersion ϕ *homogeneous*.

The above considerations allow us to construct several examples of even dimensional improper affine spheres. We shall start with a constant linear transformation A and then construct maps x and c satisfying equation (1.2). Then we shall define f by equation (1.1) to obtain the improper affine sphere.

There are two distinguished examples of this construction that can be regarded as natural generalizations of the 2-dimensional improper affine spheres. They are obtained by considering the linear transformations whose $(2n) \times (2n)$ matrices are k_n and j_n given by

$$k_{2n} = \begin{bmatrix} -i_n & 0 \\ 0 & i_n \end{bmatrix}; \quad j_{2n} = \begin{bmatrix} 0 & i_n \\ -i_n & 0 \end{bmatrix}, \quad (1.3)$$

where i_n denotes the $n \times n$ identity matrix. The k_n case is a generalization of the 2-dimensional improper affine spheres with indefinite metric ([5]). The maps x and c are the centers and mid-chords of a pair of Lagrangian submanifolds. The j_n case is a generalization of the improper affine spheres with definite metric ([6],[1],[9]). The improper affine spheres of type j_n have already been considered in connection with special Kähler manifolds ([3]), and they are sometimes called special improper, or parabolic, affine spheres. Along the paper, we shall prove many properties of these two classes of improper affine spheres, but also describe other examples of homogeneous improper affine spheres.

In dimension 2, any improper affine sphere admits a reparameterization by homogeneous coordinates: In the indefinite metric case, one can obtain asymptotic coordinates while in the definite metric case one can reparameterize the improper affine sphere with isothermal coordinates. Moreover, all examples discussed in this paper admit homogeneous reparameterizations. So we pose the following question: Are there any improper affine spheres that do not admit homogeneous reparameterizations?

The paper is organized as follows: In section 2 we define the notion of homogeneous coordinates. In section 3 we consider the main examples of the paper and prove some of its properties. In section 4 we describe some different examples. In section 5 we return to the main examples but looking at the corresponding Lagrangian immersion.

2. Even dimensional graph immersions

Let ω be the canonical symplectic form in \mathbb{R}^{2n} . Denoting $x = (x_1, \dots, x_{2n})$, we write

$$\omega = \sum_{i=1}^n dx_i \wedge dx_{i+n}.$$

So

$$\omega(v_1, v_2) = v_1^t \cdot j_{2n} \cdot v_2,$$

for any v_1, v_2 column vectors of dimension $2n$, where j_{2n} is given by (1.3).

2.1. Graph connection and its dual

Consider an immersion $\phi : M^{2n} \rightarrow \mathbb{R}^{2n} \times \mathbb{R}$ transversal to $\xi = (0, 1)$. For $r \in M$, write

$$D\phi_* X \phi_* Y = \phi_*(\nabla_X Y) + h(X, Y)\xi, \quad (2.1)$$

for any $X, Y \in T_r M$.

Lemma 2.1. *∇ is a torsion free affine connection and h is a symmetric bilinear form. We call ∇ the connection and h the metric of the graph immersion.*

Proof. Interchanging the roles of X and Y in (2.1) and subtracting we obtain

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0$$

and

$$h(X, Y) - h(Y, X) = 0.$$

□

Denote $\phi = (x, f)$ and let c be the symplectic gradient of $f : M \rightarrow \mathbb{R}$, i.e.,

$$Df(r)u = \omega(Dx(r)u, c), \quad u \in T_r M. \quad (2.2)$$

Write

$$x_{r_k r_i} = \sum_l \Gamma_{ik}^l x_{r_l}, \quad (2.3)$$

Next lemma describes the metric and connection of the graph immersion in a basis $\{e_i\}_{1 \leq i \leq 2n}$ of $T_r M$.

Lemma 2.2. *Denote $x_{r_i} = Dx(r) \cdot e_i$. We have that*

$$h(e_i, e_j) = \omega(c_{r_j}, x_{r_i}) = \omega(c_{r_i}, x_{r_j}) \quad (2.4)$$

and

$$\nabla_{e_k} e_i = \sum_l \Gamma_{ik}^l e_l.$$

Proof. Since

$$\phi_{r_i} = (x_{r_i}, \omega(c, x_{r_i})).$$

we obtain

$$\phi_{r_i r_j} = (x_{r_i r_j}, \omega(c, x_{r_i r_j})) + (0, \omega(c_{r_j}, x_{r_i})). \quad (2.5)$$

Now observe that the first parcel in the second member is tangent while the second parcel is a multiple of ξ . Thus the lemma is proved. \square

Assume that M^{2n} is simply connected and define $g : M^{2n} \rightarrow \mathbb{R}$ by

$$Dg(r)u = \omega(Dc(r)u, x), \quad u \in T_r M. \quad (2.6)$$

Equation (2.4) implies that g is well-defined. The immersion $\psi(r) = (c(r), g(r))$ is called the *dual immersion* and the function $g(c)$ is the *Legendre transform* of $f(x)$. Denoting by $\bar{\nabla}$ and \bar{h} the connection and metric of the dual immersion, we have that $\bar{h} = h$ and

$$\bar{\nabla}_{e_k} e_j = \sum_l \bar{\Gamma}_{kj}^l e_l, \quad (2.7)$$

where

$$c_{r_k r_j} = \sum_l \bar{\Gamma}_{kj}^l c_{r_l}. \quad (2.8)$$

Lemma 2.3. $\bar{\nabla}$ is the h -dual metric of ∇ . In other words, the connection $\hat{\nabla}$ of the metric h is given by

$$\hat{\nabla} = \frac{\nabla + \bar{\nabla}}{2}.$$

Proof. We must prove that

$$\frac{\partial}{\partial r_k} h(e_i, e_j) = h(\nabla_{e_k} e_i, e_j) + h(e_i, \bar{\nabla}_{e_k} e_j). \quad (2.9)$$

The first member of (2.9) is equal to

$$\begin{aligned} \omega(x_{r_k r_i}, c_{r_j}) + \omega(x_{r_i}, c_{r_k r_j}) &= \omega\left(\sum_l \Gamma_{ik}^l x_{r_l}, c_{r_j}\right) + \omega\left(x_{r_i}, \sum_l \bar{\Gamma}_{kj}^l c_{r_l}\right) \\ &= \omega(x_*(\nabla_{e_k} e_i), c_* e_j) + \omega(x_* e_i, c_*(\bar{\nabla}_{e_k} e_j)), \end{aligned}$$

which is exactly the second member of (2.9). \square

2.2. Improper affine spheres

Denote by $A(r) : T_r M \rightarrow T_r M$ the invertible linear map satisfying the condition $Dc(r) = Dx(r) \cdot A(r)$ and let $a = a(r)$ be the matrix of A in the canonical coordinates. We can write

$$c_{r_i} = \sum_{l=1}^{2n} a_{li} x_{r_l}. \quad (2.10)$$

Let

$$B(r) = Dx(r) \cdot A(r) \cdot Dx(r)^{-1} \quad (2.11)$$

and denote by $b = b(r)$ the matrix of B in the canonical basis.

The condition $c_{r_i r_j} = c_{r_j r_i}$ can be written as

$$\sum_{k=1}^{2n} a_{ki} x_{r_k r_j} = \sum_{k=1}^{2n} a_{kj} x_{r_k r_i}, \quad (2.12)$$

while the symmetry of the hessian matrix of f can be written as

$$\omega(Dx(r) \cdot u_1, Dx(r) \cdot A \cdot u_2) = \omega(Dx(r) \cdot u_2, Dx(r) \cdot A \cdot u_1), \quad (2.13)$$

for any $r \in M$, $u_1, u_2 \in T_r M$. This is equivalent to

$$\omega(v_1, B \cdot v_2) = \omega(v_2, B \cdot v_1), \quad (2.14)$$

for any $v_1, v_2 \in \mathbb{R}^{2n}$, which is the condition $b \in sp(2n)$, the Lie algebra of the symplectic group.

Lemma 2.4. *We have that*

$$\det(h) = \det(Dx)^2 \det(A).$$

Proof. The symmetric matrix h has entries

$$h_{ij} = \omega(Dx(r) \cdot e_i, Dx(r) \cdot A \cdot e_j).$$

Since $B = Dx(r) \cdot A \cdot Dx(r)^{-1}$, we have that $\det(B) = \det(A)$ and

$$h_{ij} = \omega(Dx(r) \cdot e_i, B \cdot Dx(r) \cdot e_j).$$

In terms of matrices, $h = Dx(r)^t \cdot j_{2n} \cdot b \cdot Dx(r)$. Hence

$$\det(h) = \det(Dx)^2 \det(b),$$

thus proving the lemma. \square

Corollary 2.5. *The metric h is non-degenerate if and only if A is invertible.*

Theorem 2.6. *ϕ is an improper affine sphere if and only if $\det(a)$ is constant.*

Proof. The immersion ϕ is an improper affine sphere if and only if the metric volume in the tangent space is the same as the volume determined by ξ (see [10]). The metric volume is $\sqrt{\det(h)}$, while the volume determined by ξ is

$$\det(\phi_{r_1}, \dots, \phi_{r_{2n}}, \xi) = \det(x_{r_1}, \dots, x_{r_{2n}}) = \det(Dx).$$

Thus ϕ is an improper affine sphere if and only if $\sqrt{\det(h)} = k \det(Dx)$, for some constant k . Since

$$\sqrt{\det(h)} = \det(Dx) \sqrt{\det(a)},$$

the theorem is proved. \square

2.3. Improper affine spheres with homogeneous coordinates

Denote by a^{ij} the entries of a^{-1} , the inverse of a . Then

$$x_{r_j} = \sum_{l=1}^{2n} a^{lj} c_{r_l}. \quad (2.15)$$

Lemma 2.7. *The following conditions are equivalent:*

1. *The entries a_{ij} of the matrix a are constant.*
2. *The dual connection is given by*

$$\overline{\nabla} = a^{-1} \nabla a.$$

Proof. We have that

$$c_{r_i r_j} = \sum_l a_{lj} x_{r_i r_l} + \sum_l (a_{lj})_{r_i} x_{r_l}. \quad (2.16)$$

Straightforward calculations show that the first parcel of the second member of (2.16) is equal to

$$\sum_t \left(\sum_l \sum_k a_{lj} \Gamma_{il}^k a^{tk} \right) c_{r_t}.$$

Assume that (1) holds. Then the second parcel of the second member of (2.16) is zero and thus

$$\overline{\Gamma}_{ij}^t = \sum_l \sum_k a_{lj} \Gamma_{il}^k a^{tk}.$$

This implies (2). On the other hand, if (2) holds then the second parcel of the second member of (2.16) is zero, which implies (1). \square

The image of a graph immersion $\phi(M)$ is a $2n$ -dimensional manifold in \mathbb{R}^{2n+1} and we can consider the immersion ϕ as coordinates in $\phi(M)$. When the immersion satisfies the conditions of lemma 2.7 we shall say that the coordinates ϕ in $\phi(M)$ are *homogeneous*. The following corollary is immediate from theorem 2.6:

Corollary 2.8. *An even dimensional homogeneous graph immersion is necessarily an improper affine sphere.*

It is well known that for $n = 1$ the reciprocal of the above corollary holds: Any 2-dimensional improper affine sphere admits homogeneous coordinates. More precisely, any 2-dimensional improper affine sphere with definite metric admits isothermal coordinates while any improper affine sphere with indefinite metric admits asymptotic coordinates. We formulate as a question whether or not every improper affine sphere admits a homogeneous reparameterization.

3. Two distinguished classes of improper affine spheres

In this section we shall describe in detail two classes of improper affine spheres. The first one is obtained by taking x as the center and c as the mid-chord of a pair of points of a given pair of Lagrangian submanifolds ([7]). It is a natural generalization of the class of 2-dimensional improper affine spheres with indefinite metric.

The second class is a natural generalization of the class of 2-dimensional improper affine spheres with definite metric. In [3] (see also [4]), it is proved that there is a one to one correspondence between improper affine spheres of this class, called special improper (parabolic) affine spheres, and special Kähler manifolds.

3.1. Center-chord transform of a product of Lagrangian submanifolds

Consider the following construction. Let $\alpha : U_1 \subset \mathbb{R}^n \rightarrow \mathbb{R}^{2n}$ and $\beta : U_2 \subset \mathbb{R}^n \rightarrow \mathbb{R}^{2n}$ denote Lagrangian submanifolds and assume $U_1 \times U_2 \subset \mathbb{R}^{2n}$ simply connected. Define the center $x : U_1 \times U_2 \rightarrow \mathbb{R}^{2n}$ by

$$x(s, t) = \frac{1}{2} (\alpha(s) + \beta(t))$$

and the half-chord $c : U_1 \times U_2 \rightarrow \mathbb{R}^{2n}$ by

$$c(s, t) = \frac{1}{2} (\beta(t) - \alpha(s)),$$

where $s = (s_1, \dots, s_n) \in U_1$ and $t = (t_1, \dots, t_n) \in U_2$. Observe that since α and β are Lagrangian,

$$\omega(x_{s_i}, c_{s_j}) = \omega(x_{t_i}, c_{t_j}) = 0.$$

Moreover,

$$\omega(x_{s_i}, c_{t_j}) = \omega(\alpha_{s_i}, \beta_{t_j}) = \omega(\beta_{t_j}, -\alpha_{s_i}) = \omega(x_{t_j}, c_{s_i}),$$

which implies in the existence of some function $f : U_1 \times U_2 \rightarrow \mathbb{R}$ satisfying

$$f_{s_i} = \omega(c, x_{s_i}), \quad f_{t_i} = \omega(c, x_{t_i}),$$

for any $1 \leq i \leq n$. The function $f(s, t)$ can be geometrically interpreted as the area of any 2-surface whose boundary is the chord $\beta(t) - \alpha(s)$, one curve γ_1 contained in the Lagrangian submanifold α , another curve γ_2 contained in the Lagrangian submanifold β and a fixed chord $\beta(t_0) - \alpha(s_0)$. Observe that the Lagrangian condition for α and β implies that this area does not depend on the particular choice of γ_1 and γ_2 .

One can easily verify that the immersion $\phi = (x, f)$ is a homogeneous graph immersion with matrix a given by

$$k_{2n} = \begin{bmatrix} -i_n & 0 \\ 0 & i_n \end{bmatrix}.$$

Thus ϕ is an improper affine sphere.

Next we show that the condition $a(r)^2 = i_{2n}$ implies that the immersion admits a homogeneous reparameterization. We begin with the following lemma:

Lemma 3.1. *$a(r)^2 = i_{2n}$ if and only if $a(r)$ is similar to k_{2n} .*

Proof. It is clear that $a(r)$ similar to k_n implies $a(r)^2 = i_{2n}$. Assume now that $a(r)^2 = i_{2n}$. Then the eigenvalues of $a(r)$ are ± 1 . Since $(a(r) - id_{2n}) \cdot (a(r) + id_{2n}) = 0$, the minimal polynomial of $a(r)$ contains only linear factors. Hence $a(r)$, and from equation (2.11) also $b(r)$, are diagonalizable.

Since $b(r) \in sp(2n)$, $j_{2n} \cdot b = -b^t \cdot j_{2n}$. Take u an eigenvector associated with the eigenvalue -1 . Then $b^t \cdot j_{2n}u = j_{2n}u$ and so $j_{2n}u$ is an eigenvector associated with the eigenvalue $+1$. We conclude that the dimensions of the -1 and 1 eigenspaces are equal. Hence $b(r)$, and thus also $a(r)$, are similar to k_{2n} . \square

Proposition 3.2. *Consider a graph immersion $\phi : U \subset \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n+1}$ such that the matrix $a(r)$ satisfies $a(r)^2 = i_{2n}$, for any $r \in U$. Then we can reparameterize ϕ by a homogeneous graph immersion.*

Proof. By the above lemma, $b(r)$ is similar to k_{2n} . Denote by E_1 the -1 -eigenspace and by E_2 the 1 -eigenspace. Let $\pi_1 : x(U) \rightarrow \mathbb{R}^{2n}$ be defined as $\pi_1(x) = x + c(x)$. Then, for any $v_1 \in E_1$, $v_2 \in E_2$,

$$D\pi_1(x)v_1 = v_1 + Dc(x)v_1 = 0; \quad D\pi_1(x)v_2 = v_2 + Dc(x)v_2 = 2v_2.$$

Thus ϕ_1 has rank n at all points. Denoting $\alpha = \pi_1(U)$, observe that the tangent space to α at $\pi_1(x)$ is exactly E_1 . For $v_1, w_1 \in E_1$,

$$\omega(v_1, w_1) = -\omega(v_1, k_n w_1) = -\omega(w_1, k_n v_1) = \omega(w_1, v_1).$$

Thus $\omega(v_1, w_1) = 0$ and we conclude that α is Lagrangian. Now consider $\pi_2 : x(U) \rightarrow \mathbb{R}^{2n}$ be defined as $\pi_2(x) = x - c(x)$. As above, π_2 has rank n and the tangent space to $\beta = \pi_2(U)$ at $\pi_2(x)$ equals E_2 . Moreover, β is Lagrangian.

Let $\alpha : U_1 \subset \mathbb{R}^n \rightarrow \mathbb{R}^{2n}$ and $\beta : U_2 \subset \mathbb{R}^n \rightarrow \mathbb{R}^{2n}$ be parameterizations of α and β . If $\pi_1(x) = \alpha(s)$ and $\pi_2(x) = \beta(t)$, then

$$x = \frac{1}{2}(\alpha(s) + \beta(t)); \quad c(x) = \frac{1}{2}(\beta(t) - \alpha(s)).$$

It is clear that the above defined map $x : U_1 \times U_2 \rightarrow \mathbb{R}^{2n}$ is a diffeomorphism and that $\psi(s, t) = (x(s, t), f(x(s, t)))$ is a homogeneous reparameterization of ϕ . \square

Remark 3.3. In case $n = 1$, it is well-known that we can reparameterize any improper affine sphere with indefinite metric by *asymptotic coordinates* ([5]). This result can also be obtained as a corollary of proposition 3.2. In fact, an improper affine sphere $\phi : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ with indefinite metric necessarily satisfies $\det(a)(r) = -1$, for any $r \in U$. Thus a must be equivalent to k_2 . Then proposition 3.2 implies then that we can reparameterize ϕ by homogeneous coordinates (s, t) , which in this case are called asymptotic.

3.2. Special improper affine spheres

Let $U \subset \mathbb{R}^{2n}$ be simply connected and consider a map $x : U \rightarrow \mathbb{R}^{2n}$ satisfying

$$\omega(x_{s_i}, x_{s_j}) = \omega(x_{t_i}, x_{t_j}); \quad \omega(x_{s_i}, x_{t_j}) = -\omega(x_{t_i}, x_{s_j}) \quad (3.1)$$

and

$$x_{s_i s_j} + x_{t_i t_j} = 0; \quad x_{s_i t_j} = x_{s_j t_i}. \quad (3.2)$$

From equation (3.2), we can assure the existence of $c : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ such that

$$c_{t_i} = x_{s_i}, \quad c_{s_i} = -x_{t_i}. \quad (3.3)$$

Then equation (3.1) says that there exists $f : U \rightarrow \mathbb{R}$ satisfying

$$f_{s_i} = \omega(c, x_{s_i}); \quad f_{t_i} = \omega(c, x_{t_i}). \quad (3.4)$$

It is clear that $dc = j_{2n} \cdot dx$.

It is interesting to describe this construction in terms of the complex variables (z, \bar{z}) , $z = s + it$, $\bar{z} = s - it$. Let

$$\alpha(z, \bar{z}) = x(z, \bar{z}) + ic(z, \bar{z}); \quad \beta(z, \bar{z}) = x(z, \bar{z}) - ic(z, \bar{z}). \quad (3.5)$$

Then equation (3.3) says that $\alpha_{\bar{z}} = \beta_z = 0$ and so $\alpha = \alpha(z)$, $\beta = \beta(\bar{z})$. Equation (3.1) says that the submanifolds α and β are Lagrangian. Finally

$$x(z, \bar{z}) = \frac{1}{2} (\alpha(z) + \beta(\bar{z})); \quad c(z, \bar{z}) = \frac{i}{2} (\beta(\bar{z}) - \alpha(z)).$$

We conclude that this immersion has the same structure as the one constructed in section 3.1, substituting the real variables (s, t) by the complex variables (z, \bar{z}) .

Lemma 3.4. $a(r)^2 = -i_{2n}$ if and only if $a(r)$ is similar to j_{2n} .

Proof. Analogous to lemma 3.1. □

Proposition 3.5. Consider a graph immersion $\phi : U \subset \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n+1}$ such that the matrix $a(r)$ is equivalent to j_n , for any $r \in U$. Then we can reparameterize ϕ by a homogeneous graph immersion.

Proof. Analogous to proposition 3.2 using the complex variables (z, \bar{z}) . □

Remark 3.6. In case $n = 1$, it is well-known that we can reparameterize any improper affine sphere with definite metric by *isothermal coordinates* ([6],[1],[9]). This result can also be obtained as a corollary of proposition 3.5. In fact, an improper affine sphere $\phi : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ with definite metric necessarily satisfies $\det(a)(r) = 1$, for any $r \in U$. Thus a must be equivalent to j_2 . Then proposition 3.5 implies then that we can reparameterize ϕ by homogeneous coordinates (s, t) , which in this case are called isothermal.

3.3. Parallelism with respect to the metric connection

Denote by K_{2n} and J_{2n} the linear maps of \mathbb{R}^{2n} whose matrices in the canonical basis are k_{2n} and j_{2n} , respectively. Observe that $K_{2n}^2 = I_{2n}$ and $J_{2n}^2 = -I_{2n}$, where I_{2n} denotes the identity map. We show in this section that, in these particular cases, the tensor A (equal to K_{2n} or J_{2n}) is parallel with respect to the metric connection $\hat{\nabla}$.

Proposition 3.7. *For $A = K_{2n}$ or $A = J_{2n}$ we have that $\hat{\nabla}A = 0$.*

Proof. We have that

$$\begin{aligned} (\hat{\nabla}_X A)Y &= \hat{\nabla}_X(AY) - A\hat{\nabla}_X Y \\ &= \frac{1}{2}(\nabla_X(AY) + \bar{\nabla}_X(AY) - A\nabla_X Y - A\bar{\nabla}_X Y) \\ &= \frac{1}{2}(\nabla_X(AY) + A^{-1}\nabla_X(A^2Y) - A\nabla_X Y - \nabla_X(AY)) \\ &= \frac{1}{2}(A^{-1}\nabla_X(A^2Y) - A\nabla_X Y) \end{aligned}$$

If $A^2 = -I$, then $A^{-1} = -A$ and thus this last expression vanishes. If $A^2 = I$, then $A^{-1} = A$ and this expression also vanishes. \square

4. Other examples of homogeneous graph immersions

In this section we construct homogeneous graph immersions distinct from that of section 3. From corollary 2.8, they are examples of improper affine spheres.

Given a constant $(2n) \times (2n)$ matrix a , we shall consider maps $x : U \subset \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$, U simply connected, satisfying equations (2.12) and (2.14). Equation (2.12) implies that there exists $c : U \rightarrow \mathbb{R}^{2n}$ such that $Dc(r) = Dx(r) \cdot A$. Condition (2.14) implies in the existence of $f : U \rightarrow \mathbb{R}$ whose symplectic gradient is c . It is easy to verify that the immersion $\phi(r) = (x(r), f(r))$ is homogeneous.

Example 1. The easiest example is obtained by considering $x = I_{2n}$ and $A \in sp(2n)$. It is easy to verify that conditions (2.12) and (2.14) are satisfied. This example is related to quadratic hamiltonians ([2], appendix 6).

Example 2. If one considers the product of an improper affine sphere of type k_{2n} with another of type j_{2m} one obtains a new improper affine sphere.

Example 3. Consider $n = 2$ and

$$a = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Then choose $x(s_1, s_2, t_1, t_2) = \alpha(s_1, s_2) + \beta(t_1, t_2)$ such that α and β are Lagrangian immersions and $\alpha_{s_1 s_1} = \beta_{t_2 t_2} = 0$. For example, one can take

$$\begin{aligned}\alpha(s) &= (s_1 \alpha_1(s_2) + \alpha_2(s_2), 0) \\ \beta(t) &= (0, t_2 \beta_1(t_1) + \beta_2(t_1))\end{aligned}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are arbitrary planar curves. The symplectic gradient c is given by

$$c(s_1, s_2, t_1, t_2) = -\alpha(s_1, s_2) - \int \alpha_1(s_2) ds_2 + \beta(t_1, t_2) + \int \beta_1(t_1) dt_1.$$

5. Lagrangian immersions

Let Ω be the symplectic form in \mathbb{R}^{4n} given by

$$\Omega((v_1, w_1), (v_2, w_2)) = \omega(v_1, w_2) - \omega(v_2, w_1).$$

To each immersion $\phi : M^{2n} \rightarrow \mathbb{R}^{2n} \times \mathbb{R}$, $\phi(r) = (x(r), f(r))$, we can associate the Ω -Lagrangian immersion $L : M^{2n} \rightarrow \mathbb{R}^{4n}$ $L(r) = (x(r), c(r))$, where c is the symplectic gradient of f . Reciprocally, assuming M^{2n} simply connected, if $L = (x, c)$ is a Ω -Lagrangian immersion, we can find f whose symplectic gradient is c , and f is defined up to a constant.

In this section we shall study the Lagrangian immersions associated with the examples of section 3.

5.1. Lagrangian characterization of the improper affine spheres of type k_n

Define the involution $\mathcal{K}_{4n} : \mathbb{R}^{4n} \rightarrow \mathbb{R}^{4n}$ by

$$\mathcal{K}_{4n}(v_1, v_2) = (v_2, v_1)$$

and the symplectic form Ω_k by

$$\Omega_k(v, w) = \Omega(\mathcal{K}_{4n}v, w).$$

Proposition 5.1. *Consider a Ω -Lagrangian immersion L and denote by $V(r)$ the image of $T_r M$ by the linear map $DL(r)$. The following statements are equivalent:*

1. $V(r)$ is Ω_k -Lagrangian, for any $r \in M$.
2. $V(r)$ is \mathcal{K}_{4n} -invariant, for any $r \in M$.
3. There exists a homogeneous reparameterization of $\phi = (x, f)$ of type k_{2n} .

Proof. We start with (1) implies (2). Assume that $\Omega_k(v, w) = 0$ for any $v, w \in V(r)$, and fix $v_0 \in V(r)$. Then

$$\Omega(v + \lambda \mathcal{K}_{4n} v_0, w + \mu \mathcal{K}_{4n} v_0) = \lambda \Omega_k(v_0, v) - \mu \Omega_k(v_0, w) = 0.$$

Thus $\text{span}\{V(r), \mathcal{K}_{4n} v_0\}$ is Ω -Lagrangian and thus $\mathcal{K}_{4n} v_0 \in V(r)$.

Assuming that $V(r)$ is \mathcal{K}_{4n} -invariant, we have that, for any $v, w \in V(r)$,

$$\Omega_k(v, w) = \Omega(\mathcal{K}_{4n}v, w) = 0.$$

Thus (2) implies (1).

We shall now prove that (3) implies (2). Take $v = Dx(r) \cdot u$, $w = Dx(r) \cdot k_n u$. Then

$$\mathcal{K}_{4n}(v, w) = (Dx(r) \cdot k_n u, Dx(r) \cdot u) = (Dx(r) \cdot u_1, Dx(r) \cdot k_n u_1)$$

also belongs to $V(r)$.

Finally let us prove that (2) implies (3). Since $V(r)$ is \mathcal{K}_{4n} -invariant,

$$\mathcal{K}_{4n}(Dx(r) \cdot u, Dx(r) \cdot a(r)u) = (Dx(r) \cdot a(r)u, Dx(r) \cdot u)$$

belong to $V(r)$, for any $u \in T_r M$. Thus $a(r)^2 = i_{2n}$. Now proposition 3.2 implies that the improper affine sphere admits a homogeneous reparameterization. \square

5.2. Lagrangian characterization of the improper affine spheres of type j_n

Consider the complex structure $J_{4n} : \mathbb{R}^{4n} \rightarrow \mathbb{R}^{4n}$ by

$$J_{4n}(v_1, v_2) = (v_2, -v_1)$$

and the symplectic form Ω_j by

$$\Omega_j(v, w) = \Omega(J_{4n}v, w).$$

Proposition 5.2. *Consider a Ω -Lagrangian immersion L and denote by $V(r)$ the image of $T_r M$ by the linear map $DL(r)$. The following statements are equivalent:*

1. $V(r)$ is Ω_j -Lagrangian, for any $r \in M$.
2. $V(r)$ is J_{4n} -invariant, for any $r \in M$.
3. There exists a homogeneous reparameterization of $\phi = (x, f)$ of type j_{2n} .

Proof. Similar to proposition 5.1. \square

References

- [1] Aledo, J.A., Chaves, R.M.B., Gálvez, J.A.: *The Cauchy Problem for Improper Affine Spheres and the Hessian One Equation*. Trans. Amer. Math. Soc. 359 (9), 4183-4208, 2007.
- [2] Arnold, V.I.: *Mathematical methods of classical mechanics*. Springer-Verlag, 1974.
- [3] Baues, O., Cortés, V.: *Realisation of special Kähler manifolds as parabolic spheres*, Proc. Amer. math. Soc. 129(8), 2403-2407, 2000.
- [4] Cortés, V.: *A holomorphic representation formula for parabolic hyperspheres*, arXiv: math/0107037.

- [5] Craizer, M., da Silva, M.A.H.B., Teixeira, R.C.: *A geometric representation of improper indefinite affine spheres with singularities*, Journal of Geometry, 100(1), 65-78, 2011.
- [6] Craizer, M.: *Singularities of convex improper affine maps*, Journal of Geometry, 103(2), 207-217, 2012.
- [7] Domitrz, W., Rios, P. de M.: *Singularities of equidistants and global symmetry sets of Lagrangian submanifolds*, arXiv: 1007.1939.
- [8] Loftin, J.: *Survey on affine spheres*. Handbook of Geometric Analysis, n.2, Adv.Lect.Math.(13), International Press, 2010.
- [9] Martinez, A.: *Improper affine maps*. Mathematische Zeitschrift **249**, 755-766, 2005.
- [10] Nomizu, K., Sasaki, T.: *Affine Differential Geometry*. Cambridge University Press, 1994.

Marcos Craizer
Departamento de Matemática - PUC-Rio
Rio de Janeiro
Brazil

e-mail: craizer@puc-rio.br

Pedro de M. Rios
Departamento de Matemática - ICMC
Universidade de São Paulo
São Carlos
Brazil

e-mail: prios@icmc.usp.br